Chapter
5

Permutations and Combinations

Contents

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The concepts of permutations and combinations can be traced back to the advent of Jainism in India and perhaps even earlier. Among the Jains, Mahavira, *(around 850 A.D.) is perhaps the world's first mathematician credited with providing the general formulae for permutations and combinations.*

Bhaskaracharya *(born* 1114 A.D.*) treated the subject matter of permutations and combinations under the name* Anka Pasha *in his famous work* Lilavati. *In addition to the general formulae for ^nC_r and ^nP_r already provided by* Mahavira,

Outside India, the subject matter of permutations and combinations had its humble beginnings in China in the famous book I-King (Book of changes). The first book which gives a comkplete treatment of the subject matter of permutations and combinations is Ars conjectandi *written by a Swiss,* Jacob Bernouli *(1654-1705 A.D.) posthumously published in* 1713 A.D. *This book contains essentially the theory of permutations and combinations as is known today.*

Permutations and Combinations

Permutations

5.1 The Factorial

Factorial notation: Let n be a positive integer. Then, the continued product of first n natural numbers is called factorial $\lfloor n$, to be denoted by n ! or $\lfloor n$. Also, we define 0 ! = 1.

when n is negative or a fraction, n ! is not defined.

Thus, $n! = n(n-1)(n-2)......3.2.1.$

Deduction: $n! = n(n-1)(n-2)(n-3)......3.2.1$

$$= n[(n-1)(n-2)(n-3)......3.2.1] = n[(n-1)!]$$

Thus, $5! = 5\times(4!), 3! = 3\times(2!)$ and $2! = 2\times(1!)$

Also, $1! = 1\times(0!) \Rightarrow 0! = 1$.

5.2 Exponent of Prime *p* in *n* !

Let p be a prime number and n be a positive integer. Then the last integer amongst 1, 2, 3, ...$(n-1)$, n which is divisible by p is $\left[\frac{n}{p}\right]p$, where $\left[\frac{n}{p}\right]$ denote the greatest integer less than or equal to $\frac{n}{p}$.

For example: $\left[\frac{10}{3}\right]=3$, $\left[\frac{12}{5}\right]=2$, $\left[\frac{15}{3}\right]=5$ etc.

Let $E_p(n)$ denotes the exponent of the prime p in the positive integer n. Then,

$$E_p(n!) = E_p(1.2.3..........(n-1)n) = E_p\left(p.2p.3p.......\left[\frac{n}{p}\right]p\right) = \left[\frac{n}{p}\right]+E_p\left(1.2.3......\left[\frac{n}{p}\right]\right)$$

[$\because$ Remaining integers between 1 and n are not divisible by p]

Now the last integer amongst 1, 2, 3,..... $\left[\frac{n}{p}\right]$ which is divisible by p is

$\left[\frac{n/p}{p}\right] = \left[\frac{n}{p^2}\right] = \left[\frac{n}{p}\right] + E_p\left(p, 2p, 3p....\left[\frac{n}{p^2}\right]p\right)$ because the remaining natural numbers from 1 to $\left[\frac{n}{p}\right]$ are not divisible by p = $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+E_p\left(1.2.3......\left[\frac{n}{p^2}\right]\right)$

Similarly we get $E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots \left[\frac{n}{p^S}\right]$

where S is the largest natural number. Such that $p^S \le n < p^{S+1}$.

5.3 Fundamental Principles of Counting

(1) **Addition principle** : Suppose that A and B are two disjoint events (mutually exclusive); that is, they never occur together. Further suppose that A occurs in m ways and B in n ways. Then A or B can occur in $m + n$ ways. This rule can also be applied to more than two mutually exclusive events.

Example: 1 A college offers 7 courses in the morning and 5 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening

(a) 27 (b) 15 (c) 12 (d) 35

Solution: (c) The student has seven choices from the morning courses out of which he can select one course in 7 ways.

For the evening course, he has 5 choices out of which he can select one course in 5 ways.

Hence he has total number of 7 + 5 = 12 choices.

(2) **Multiplication principle** : Suppose that an event X can be decomposed into two stages A and B. Let stage A occur in m ways and suppose that these stages are unrelated, in the sense that stage B occurs in n ways regardless of the outcome of stage A. Then event X occur in mn ways. This rule is applicable even if event X can be decomposed in more than two stages.

Note : ❑ The above principle can be extended for any finite number of operations and may be stated as under :

If one operation can be performed independently in m different ways and if second operation can be performed independently in n different ways and a third operation can be performed independently in p different ways and so on, then the total number of ways in which all the operations can be performed in the stated order is $(m \times n \times p \times \ldots\ldots)$

Example: 2 In a monthly test, the teacher decides that there will be three questions, one from each of exercise 7, 8 and 9 of the text book. If there are 12 questions in exercise 7, 18 in exercise 8 and 9 in exercise 9, in how many ways can three questions be selected

(a) 1944 (b) 1499 (c) 4991 (d) None of these

Solution: (a) There are 12 questions in exercise 7. So, one question from exercise 7 can be selected in 12 ways. Exercise 8 contains 18 questions. So, second question can be selected in 18 ways. There are 9 questions in exercise 9. So, third question can be selected in 9 ways. Hence, three questions can be selected in 12 × 18 × 9 = 1944 ways.

5.4 Definition of Permutation

The ways of arranging or selecting a smaller or an equal number of persons or objects at a time from a given group of persons or objects with due regard being paid to the order of arrangement or selection are called the (different) *permutations*.

For example : Three different things *a, b* and *c* are given, then different arrangements which can be made by taking two things from three given things are *ab, ac, bc, ba, ca, cb*.

Therefore the number of permutations will be 6.

5.5 Number of Permutations without Repetition

(1) Arranging n objects, taken r at a time equivalent to filling r places from n things

r-places :	1	2	3	4		r
Number of choices :	n	$(n-1)$	$(n-$	$)(n-$		$n-(r-$

The number of ways of arranging = The number of ways of filling r places.

$$= n(n-1)(n-2).......(n-r+1) = \frac{n(n-1)(n-2).....(n-r+1)((n-r)!)}{(n-r)!} = \frac{n!}{(n-r)!} = {}^nP_r$$

(2) The number of arrangements of n different objects taken all at a time = ${}^nP_n = n!$

Note:

❑ ${}^nP_0 = \frac{n!}{n!} = 1; {}^nP_r = n.{}^{n-1}P_{r-1}$

❑ $0! = 1; \frac{1}{(-r)!} = 0$ or $(-r)! = \infty$ $(r \in N)$

Example: 3 If ${}^nP_4 : {}^nP_5 = 1:2$, then $n =$ **[MP PET 1987; Rajasthan PET 1996]**

(a) 4 (b) 5 (c) 6 (d) 7

Solution: (c) $\frac{{}^nP_4}{{}^nP_5} = \frac{1}{2} \Rightarrow \frac{n!}{(n-4)!} \times \frac{(n-5)!}{n!} = \frac{1}{2} \Rightarrow n-4 = 2 \Rightarrow n = 6$.

Example: 4 In a train 5 seats are vacant then how many ways can three passengers sit **[Rajasthan PET 1985; MP PET 2003]**

(a) 20 (b) 30 (c) 60 (d) 10

Solution: (c) Number of ways are = ${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{120}{2} = 60$.

Example: 5 How many words comprising of any three letters of the word "UNIVERSAL" can be formed

(a) 504 (b) 405 (c) 540 (d) 450

Solution: (a) Required numbers of words = ${}^9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = 504$.

Example: 6 How many numbers of five digits can be formed from the numbers 2, 0, 4, 3, 8 when repetition of digit is not allowed

[MP PET 2000]

(a) 96 (b) 120 (c) 144 (d) 14

Solution: (a) Given numbers are 2, 0, 4, 3, 8

Numbers can be formed = {Total – Those beginning with 0}

= {5 ! – 4 !} = 120 – 24 = 96.

Example: 7 How many numbers can be made with the help of the digits 0, 1, 2, 3, 4, 5 which are greater than 3000 (repetition is not allowed)

(a) 180 (b) 360 (c) 1380 (d) 1500

Solution: (c) All the 5 digit numbers and 6 digit numbers are greater than 3000. Therefore number of 5 digit numbers = ${}^6P_5 - {}^5P_5 = 600$.

{Since the case that 0 will be at ten thousand place should be omit}. Similarly number of 6 digit numbers 6 ! – 5 ! = 600.

Now the numbers of 4 digit numbers which are greater than 3000, having 3, 4 or 5 at first place, this can be done in 3 ways and remaining 3 digit may be filled from remaining 5 digits *i.e.,* required number of 4 digit numbers are ${}^5P_3 \times 3 = 180$.

Hence total required number of numbers = 600 + 600 + 180 = 1380.

5.6 Number of Permutations with Repetition

(1) The number of permutations (arrangements) of n different objects, taken r at a time, when each object may occur once, twice, thrice,........upto r times in any arrangement = The number of ways of filling r places where each place can be filled by any one of n objects.

r – places :	1	2	3	4		r
Number of choices :	n	n	n	n		n

The number of permutations = The number of ways of filling r places = $(n)^r$

(2) The number of arrangements that can be formed using n objects out of which p are identical (and of one kind) q are identical (and of another kind), r are identical (and of another kind) and the rest are distinct is $\dfrac{n!}{p!q!r!}$.

Example: 8 The number of arrangement of the letters of the word "CALCUTTA" **[MP PET 1984]**

(a) 2520 (b) 5040 (c) 10080 (d) 40320

Solution: (b) Required number of ways $= \dfrac{8!}{2!2!2!} = 5040$. [since here 2C's, 2T's and 2A's]

Example: 9 The number of 5 digit telephone numbers having at least one of their digits repeated is

(a) 90,000 (b) 100,000 (c) 30,240 (d) 69,760

Solution: (d) Using the digits 0, 1, 2,.......,9 the number of five digit telephone numbers which can be formed is 10^5.

(since repetition is allowed)

The number of five digit telephone numbers which have none of the digits repeated = ${}^{10}P_5 = 30240$

$\therefore$ The required number of telephone numbers = $10^5 - 30240 = 69760$.

Example: 10 How many words can be made from the letters of the word 'COMMITTEE' **[MP PET 2002; RPET 1986]**

(a) $\dfrac{9!}{(2!)^2}$ (b) $\dfrac{9!}{(2!)^3}$ (c) $\dfrac{9!}{2!}$ (d) 9 !

Solution: (b) Number of words = $\dfrac{9!}{2!2!2!} = \dfrac{9!}{(2!)^3}$ [Since here total number of letters is 9 and 2M's, 2T's and 2E's]

5.7 Conditional Permutations

(1) Number of permutations of n dissimilar things taken r at a time when p particular things always occur $= {}^{n-p}C_{r-p}\, r!$

(2) Number of permutations of n dissimilar things taken r at a time when p particular things never occur $= {}^{n-p}C_r\, r!$

(3) The total number of permutations of n different things taken not more than r at a time, when each thing may be repeated any number of times, is $\frac{n(n^r-1)}{n-1}$.

(4) Number of permutations of n different things, taken all at a time, when m specified things always come together is $m!\times(n-m+1)!$

(5) Number of permutations of n different things, taken all at a time, when m specified things never come together is $n!-m!\times(n-m+1)!$

(6) Let there be n objects, of which m objects are alike of one kind, and the remaining $(n-m)$ objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is $\frac{n!}{(m!)\times(n-m)!}$.

Note : ❑ The above theorem can be extended further *i.e.*, if there are n objects, of which p_1 are alike of one kind; p_2 are alike of another kind; p_3 are alike of 3rd kind;......: p_r are alike of r^{th} kind such that $p_1+p_2+......+p_r=n$; then the number of permutations of these n objects is $\frac{n!}{(p_1!)\times(p_2!)\times......\times(p_r!)}$.

Important Tips

- ***Gap method :*** *Suppose 5 males A, B, C, D, E are arranged in a row as × A × B × C × D × E ×. There will be six gaps between these five. Four in between and two at either end. Now if three females P, Q,R are to be arranged so that no two are together we shall use gap method i.e., arrange them in between these 6 gaps. Hence the answer will be* $^6P_3$.
- ***Together :*** *Suppose we have to arrange 5 persons in a row which can be done in 5 ! = 120 ways. But if two particular persons are to be together always, then we tie these two particular persons with a string. Thus we have 5 – 2 + 1 (1 corresponding to these two together) = 3 +1 = 4 units, which can be arranged in 4! ways. Now we loosen the string and these two particular can be arranged in 2 ! ways. Thus total arrangements = 24 × 2 = 48.*

  Never together = Total – Together = 120 – 48 = 72.

Example: 11 All the letters of the word 'EAMCET' are arranged in all possible ways. The number of such arrangement in which two vowels are not adjacent to each other is **[EAMCET 1987; DCE 2000]**

(a) 360 (b) 114 (c) 72 (d) 54

Solution: (c) First we arrange 3 consonants in 3 ! ways and then at four places (two places between them and two places on two sides) 3 vowels can be placed in $^4P_3\times\frac{1}{2!}$ ways.

Hence the required ways = 3 ! × $^4P_3\times\frac{1}{2!}=72$.

Example: 12 The number of words which can be made out of the letters of the word 'MOBILE' when consonants always occupy odd places is

(a) 20 (b) 36 (c) 30 (d) 720

Solution: (b) The word 'MOBILE' has three even places and three odd places. It has 3 consonants and 3 vowels. In three odd places we have to fix up 3 consonants which can be done in ${}^3P_3$ ways. Now remaining three places we have to fix up remaining three places which can be done in ${}^3P_3$ ways.

The total number of ways = ${}^3P_3 \times {}^3P_3 = 36$.

Example: 13 The number of 4 digit number that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contain digit 1 is

(a) 1225 (b) 1252 (c) 1522 (d) 480

Solution: (d) After fixing 1 at one position out of 4 places, 3 places can be filled by ${}^7P_3$ ways. But some numbers whose fourth digit is zero, so such type of ways = ${}^6P_2$

$\therefore$ Total ways = ${}^7P_3 - {}^6P_2 = 480$.

Example: 14 m men and n women are to be seated in a row, so that no two women sit together. If $m > n$, then the number of ways in which they can be seated is

(a) $\frac{m!(m+1)!}{(m-n+1)!}$ (b) $\frac{m!(m-1)!}{(m-n+1)!}$ (c) $\frac{(m-1)!(m+1)!}{(m-n+1)!}$ (d) None of these

Solution: (a) First arrange m men, in a row in m ! ways. Since $n < m$ and no two women can sit together, in any one of the m ! arrangement , there are $(m + 1)$ places in which n women can be arranged in ${}^{m+1}P_n$ ways.

$\therefore$ By the fundamental theorem, the required number of arrangement = $m!\ {}^{m+1}P_n = \frac{m!(m+1)!}{(m-n+1)!}$.

Example: 15 If the letters of the word 'KRISNA' are arranged in all possible ways and these words are written out as in a dictionary, then the rank of the word 'KRISNA' is

(a) 324 (b) 341 (c) 359 (d) None of these

Solution: (a) Words starting from *A* are 5 ! = 120;
Words starting from *I* are 5 ! = 120
Words starting from *KA* are 4 ! = 24;
Words starting from *KI* are 4 ! = 24
Words starting from *KN* are 4 ! = 24;
Words starting from *KRA* are 3 ! = 6
Words starting from *KRIA* are 2 ! = 2;
Words starting from *KRIN* are 2 ! = 2
Words starting from *KRIS* are 1 ! = 1
Words starting from *KRISNA* are 1 ! = 1
Hence rank of the word KRISNA is 324

Example: 16 We are to form different words with the letters of the word 'INTEGER'. Let m_1 be the number of words in which *I* and *N* are never together, and m_2 be the number of words which begin with *I* and end with *R*. Then m_1/m_2 is equal to

[AMU 2000]

(a) 30 (b) 60 (c) 90 (d) 180

Solution: (a) We have 5 letters other than '*I*' and '*N*' of which two are identical (*E*'*s*). We can arrange these letters in a line in $\frac{5!}{2!}$ ways. In any such arrangement '*I*' and '*N*' can be placed in 6 available gaps in ${}^6P_2$ ways, so required number = $\frac{5!}{2!}{}^6P_2 = m_1$.

Now, if word start with *I* and end with *R* then the remaining letters are 5. So, total number of ways = $\frac{5!}{2!} = m_2$.

$$\therefore \frac{m_1}{m_2} = \frac{5!}{2!}.\frac{6!}{4!}.\frac{2!}{5!} = 30.$$

Example: 17 An n digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is **[IIT 1998]**

(a) 6 (b) 7 (c) 8 (d) 9

Solution: (b) Since at any place, any of the digits 2, 5 and 7 can be used total number of such positive n-digit numbers are 3^n. Since we have to form 900 distinct numbers, hence $3^n \geq 900 \Rightarrow n = 7$.

Example: 18 The number of numbers that can be formed with the help of the digits 1, 2, 3, 4, 3, 2, 1 so that odd digits always occupy odd places, is

(a) 24 (b) 18 (c) 12 (d) 30

Solution: (b) The 4 odd digits 1, 3, 3, 1 can be arranged in the 4 odd places, in $\frac{4!}{2!2!} = 6$ ways and 3 even digits 2, 4, 2 can be arranged in the three even places $\frac{3!}{2!} = 3$ ways. Hence the required number of ways = 6 × 3 = 18.

5.8 Circular Permutations

So far we have been considering the arrangements of objects in a line. Such permutations are known as linear permutations.

Instead of arranging the objects in a line, if we arrange them in the form of a circle, we call them, circular permutations.

In circular permutations, what really matters is the position of an object relative to the others.

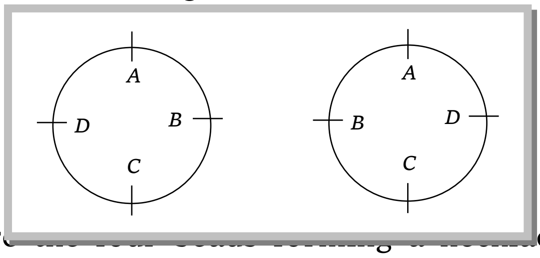
Thus, in circular permutations, we fix the position of the one of the objects and then arrange the other objects in all possible ways.

There are two types of circular permutations :

(i) The circular permutations in which clockwise and the anticlockwise arrangements give rise to different permutations, *e.g.* Seating arrangements of persons round a table.

(ii) The circular permutations in which clockwise and the anticlockwise arrangements give rise to same permutations, e.g. arranging some beads to form a necklace.

Look at the circular permutations, given below :



Suppose *A, B, C, D* ar[illegible]ce. They have been arranged in clockwise and anticlockwise directions in the first and second arrangements respectively.

Now, if the necklace in the first arrangement be given a turn, from clockwise to anticlockwise, we obtain the second arrangement. Thus, there is no difference between the above two arrangements.

(1) **Difference between clockwise and anticlockwise arrangement :** If anticlockwise and clockwise order of arrangement are not distinct *e.g.*, arrangement of beads in a necklace, arrangement of flowers in garland etc. then the number of circular permutations of n distinct items is $\frac{(n-1)!}{2}$

(2) **Theorem on circular permutations**

Theorem 1 : The number of circular permutations of n different objects is $(n-1)!$

Theorem 2 : The number of ways in which n persons can be seated round a table is $(n-1)!$

Theorem 3 : The number of ways in which n different beads can be arranged to form a necklace, is $\frac{1}{2}(n-1)!$.

Note : ❑ When the positions are numbered, circular arrangement is treated as a linear arrangement.

❑ In a linear arrangement, it does not make difference whether the positions are numbered or not.

Example: 19 In how many ways a garland can be made from exactly 10 flowers **[MP PET 1984]**

(a) 10 ! (b) 9 ! (c) 2 (9!) (d) $\frac{9!}{2}$

Solution: (d) A garland can be made from 10 flowers in $\frac{1}{2}(9!)$ ways [$\because$ n flower's garland can be made in $\frac{1}{2}(n-1)!$ ways]

Example: 20 In how many ways can 5 boys and 5 girls sit in a circle so that no boys sit together

(a) 5! × 5! (b) 4! × 5 ! (c) $\frac{5!\times 5!}{2}$ (d) None of these

Solution: (b) Since total number of ways in which boys can occupy any place is $(5-1)! = 4!$ and the 5 girls can be sit accordingly in 5! ways. Hence required number of ways are 4 ! × 5 !.

Example: 21 The number of ways in which 5 beads of different colours form a necklace is

(a) 12 (b) 24 (c) 120 (d) 60

Solution: (a) The number of ways in which 5 beads of different colours can be arranged in a circle to form a necklace are

$= (5-1)! = 4!$.

But the clockwise and anticlockwise arrangement are not different (because when the necklace is turned over one gives rise to another). Hence the total number of ways of arranging the beads = $\frac{1}{2}(4!) = 12$.

Example: 22 The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two female are not seated together is

(a) 480 (b) 600 (c) 720 (d) 840

Solution: (a) Fix up a male and the remaining 4 male can be seated in 4! ways. Now no two female are to sit together and as such the 2 female are to be arranged in five empty seats between two consecutive male and number of arrangement will be ${}^5P_2$. Hence by fundamental theorem the total number of ways is = $4! \times {}^5P_2 = 24 \times 20 = 480$ ways.

Combinations

5.9 Definition

Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a combination.

Suppose we want to select two out of three persons *A, B* and *C*.

We may choose *AB* or *BC* or *AC*.

Clearly, *AB* and *BA* represent the same selection or group but they give rise to different arrangements.

Clearly, in a group or selection, the order in which the objects are arranged is immaterial.

Notation: The number of all combinations of *n* things, taken *r* at a time is denoted by $C(n,r)$ or ${}^nC_r$ or $\binom{n}{r}$.

(1) **Difference between a permutation and combination :** (i) In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

(ii) In a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential. For example *A, B* and *B, A* are same as combination but different as permutations.

(iii) Practically to find the permutation of *n* different items, taken *r* at a time, we first select *r* items from *n* items and then arrange them. So usually the number of permutations exceeds the number of combinations.

(iv) Each combination corresponds to many permutations. For example, the six permutations *ABC, ACB, BCA, BAC, CBA* and *CAB* correspond to the same combination *ABC*.

Note : ❑ Generally we use the word 'arrangements' for permutations and word "selection" for combinations.

5.10 Number of Combinations without Repetition

The number of combinations (selections or groups) that can be formed from *n* different objects taken $r (0 \le r \le n)$ at a time is ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

Let the total number of selections (or groups) = *x*. Each group contains *r* objects, which can be arranged in *r* ! ways. Hence the number of arrangements of *r* objects = $x \times (r!)$. But the number of arrangements = ${}^nP_r$.

$$\Rightarrow x \times (r!) = {}^nP_r \Rightarrow x = \frac{{}^nP_r}{r!} \Rightarrow x = \frac{n!}{r!(n-r)!} = {}^nC_r .$$

Important Tips

- ☞ ${}^nC_r$ *is a natural number.*
- ☞ ${}^nC_0 = {}^nC_n = 1, {}^nC_1 = n$
- ☞ ${}^nC_r = {}^nC_{n-r}$
- ☞ ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ☞ ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$
- ☞ $n.{}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$
- ☞ *If n is even then the greatest value of* ${}^nC_r$ *is* ${}^nC_{n/2}$.
- ☞ *If n is odd then the greatest value of* ${}^nC_r$ *is* $\dfrac{{}^nC_{n+1}}{2}$ *or* $\dfrac{{}^nC_{n-1}}{2}$.
- ☞ ${}^nC_r = \dfrac{n}{r}.{}^{n-1}C_{r-1}$
- ☞ $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$
- ☞ ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$
- ☞ ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$

☞ $^{2n+1}C_0 + ^{2n+1}C_1 + ^{2n+1}C_2 + ... + ^{2n+1}C_n = 2^{2n}$ ☞ $^nC_n + ^{n+1}C_n + ^{n+2}C_n + ^{n+3}C_n + ... + ^{2n-1}C_n = ^{2n}C_{n+1}$

Note : ❑ Number of combinations of n dissimilar things taken all at a time $^nC_n = \frac{n!}{n!(n-n)!} = \frac{1}{0!} = 1$, $(\because 0! = 1)$.

Example: 23 If $^{15}C_{3r} = ^{15}C_{r+3}$, then the value of r is **[IIT 1967; Rajasthan PET 1991; MP PET 1998; Karnataka CET 1996]**

(a) 3 (b) 4 (c) 5 (d) 8

Solution: (a) $^{15}C_{3r} = ^{15}C_{r+3} \Rightarrow ^{15}C_{15-3r} = ^{15}C_{r+3} \Rightarrow 15 - 3r = r + 3 \Rightarrow r = 3$.

Example: 24 $\frac{^nC_r}{^nC_{r-1}} =$ **[MP PET 1984]**

(a) $\frac{n-r}{r}$ (b) $\frac{n+r-1}{r}$ (c) $\frac{n-r+1}{r}$ (d) $\frac{n-r-1}{r}$

Solution: (c) $\frac{^nC_r}{^nC_{r-1}} = \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r-1)!(n-r+1)!}} \Rightarrow \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(n-r+1)(r-1)!(n-r)!}{r(r-1)!(n-r)!} = \frac{(n-r+1)}{r}$.

Example: 25 If $^{n+1}C_3 = 2\,^nC_2$, then $n =$ **[MP PET 2000]**

(a) 3 (b) 4 (c) 5 (d) 6

Solution: (c) $^{n+1}C_3 = 2.^nC_2$

$\Rightarrow \frac{(n+1)!}{3!(n-2)!} = 2.\frac{n!}{2!(n-2)!} \Rightarrow \frac{n+1}{3.2!} = \frac{2}{2!} \Rightarrow n+1 = 6 \Rightarrow n = 5$.

Example: 26 If $^nC_{r-1} = 36, ^nC_r = 84$ and $^nC_{r+1} = 126$ then the value of r is **[IIT 1979; Pb. CET 1993; DCE 1999; MP PET 2001]**

(a) 1 (b) 2 (c) 3 (d) None of these

Solution: (c) Here $\frac{^nC_{r-1}}{^nC_r} = \frac{36}{84}$ and $\frac{^nC_r}{^nC_{r+1}} = \frac{84}{126}$

$3n - 10r = -3$ and $4n - 10r = 6$; on solving we get $n = 9$ and $r = 3$.

Example: 27 In a conference of 8 persons, if each person shake hand with the other one only, then the total number of shake hands shall be

(a) 64 (b) 56 (c) 49 (d) 28

Solution: (d) Total number of shake hands when each person shake hands with the other once only = $^8C_2 = 28$ ways.

Example: 28 How many words of 4 consonants and 3 vowels can be formed from 6 consonants and 5 vowels.**[Rajasthan PE**

(a) 75000 (b) 756000 (c) 75600 (d) None of these

Solution: (b) Required number of words = $^6C_4 \times ^5C_3 \times 7!$ = 756000

[Selection can be made in $^6C_4 \times ^5C_3$ while the 7 letters can be arranged in 7!]

Example: 29 To fill 12 vacancies there are 25 candidates of which five are from scheduled caste. If 3 of the vacancies are reserved for scheduled caste candidates while the rest are open to all, then the number of ways in which the selection can be made

[Rajasthan PET 1981]

(a) $^5C_3 \times ^{22}C_9$ (b) $^{22}C_9 - ^5C_3$ (c) $^{22}C_3 + ^5C_3$ (d) None of these

Solution: (a) The selection can be made in ${}^5C_3 \times {}^{22}C_9$ [since 3 vacancies filled from 5 candidates in ${}^5C_3$ ways and now remaining candidates are 22 and remaining seats are 9, then remaining vacancies filled by ${}^{22}C_9$ ways. Hence total number of ways ${}^5C_3 \times {}^{22}C_9$.

5.11 Number of Combinations with Repetition and All Possible Selections

(1) The number of combinations of n distinct objects taken r at a time when any object may be repeated any number of times.

= coefficient of x^r in $(1+x+x^2+.......+x^r)^n$ = coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$

(2) The total number of ways in which it is possible to form groups by taking some or all of n things at a time is 2^n-1.

(3) The total number of ways in which it is possible to make groups by taking some or all out of $n=(n_1+n_2+....)$ things, when n_1 are alike of one kind, n_2 are alike of second kind, and so on is $\{(n_1+1)(n_2+1)......\}-1$.

(4) The number of selections of r objects out of n identical objects is 1.

(5) Total number of selections of zero or more objects from n identical objects is $n+1$.

(6) The number of selections taking at least one out of $a_1+a_2+a_3+......+a_n+$ k objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on...... a_n are alike (of n[th] kind) and k are distinct = $[(a_1+1)(a_2+1)(a_3+1).......(a_n+1)]2^k-1$.

Example: 30 There are 10 lamps in a hall. Each one of them can be switched on independently. The number of ways in which the hall can be illuminated is

(a) 10^2 (b) 1023 (c) 2^{10} (d) 10 !

Solution: (b) Number of ways are = $2^{10}-1=1023$

[– 1 corresponds to none of the lamps is being switched on.]

Example: 31 10 different letters of English alphabet are given. Out of these letters, words of 5 letters are formed. How many words are formed when atleast one letter is repeated

(a) 99748 (b) 98748 (c) 96747 (d) 97147

Solution: (a) Number of words of 5 letters in which letters have been repeated any times = 10^5

But number of words on taking 5 different letters out of 10 = ${}^{10}C_5=252$

$\therefore$ Required number of words = 10^5-252 = 99748.

Example: 32 A man has 10 friends. In how many ways he can invite one or more of them to a party

(a) 10 ! (b) 2^{10} (c) $10!-1$ (d) $2^{10}-1$

Solution: (d) Required number of friend = $2^{10}-1$ (Since the case that no friend be invited *i.e.,* ${}^{10}C_0$ is excluded)

Example: 33 Numbers greater than 1000 but not greater than 4000 which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are

(a) 350 (b) 375 (c) 450 (d) 576

Solution: (b) Numbers greater than 1000 and less than or equal to 4000 will be of 4 digits and will have either 1 (except 1000) or 2 or 3 in the first place with 0 in each of remaining places.

After fixing 1[st] place, the second place can be filled by any of the 5 numbers. Similarly third place can be filled up in 5 ways and 4[th] place can be filled up in 5 ways. Thus there will be 5 × 5 × 5 = 125 ways in which 1 will be in first place but this include 1000 also hence there will be 124 numbers having 1 in the first place. Similarly 125 for each 2 or 3. One number will be in which 4 in the first place and *i.e.,* 4000. Hence the required numbers are 124 + 125 + 125 +1 = 375 ways.

5.12 Conditional Combinations

(1) The number of ways in which r objects can be selected from n different objects if k particular objects are

(i) Always included = ${}^{n-k}C_{r-k}$ (ii) Never included = ${}^{n-k}C_r$

(2) The number of combinations of n objects, of which p are identical, taken r at a time is

$= {}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + ... + {}^{n-p}C_0$ if $r \le p$ and

$= {}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + ... + {}^{n-p}C_{r-p}$ if $r > p$

Example: 34 In the 13 cricket players 4 are bowlers, then how many ways can form a cricket team of 11 players in which at least 2 bowlers included

(a) 55 (b) 72 (c) 78 (d) None of these

Solution: (c) The number of ways can be given as follows:

2 bowlers and 9 other players = ${}^4C_2 \times {}^9C_9$; 3 bowlers and 8 other players = ${}^4C_3 \times {}^9C_8$

4 bowlers and 7 other players = ${}^4C_4 \times {}^9C_7$

Hence required number of ways = 6 × 1 + 4 × 9 + 1 × 36 = 78.

Example: 35 In how many ways a team of 10 players out of 22 players can be made if 6 particular players are always to be included and 4 particular players are always excluded

(a) ${}^{22}C_{10}$ (b) ${}^{18}C_3$ (c) ${}^{12}C_4$ (d) ${}^{18}C_4$

Solution: (c) 6 particular players are always to be included and 4 are always excluded, so total number of selection, now 4 players out of 12.

Hence number of ways = ${}^{12}C_4$.

Example : 36 In how many ways can 6 persons to be selected from 4 officers and 8 constables, if at least one officer is to be included

[Roorkee 1985; MP PET 2001]

(a) 224 (b) 672 (c) 896 (d) None of these

Solution: (c) Required number of ways = ${}^4C_1 \times {}^8C_5 + {}^4C_2 \times {}^8C_4 + {}^4C_3 \times {}^8C_3 + {}^4C_4 \times {}^8C_2$ = 4 × 56 + 6 × 70 + 4 × 56 + 1 × 28 = 896.

5.13 Division into Groups

Case I : (1) The number of ways in which n different things can be arranged into r different groups is ${}^{n+r-1}P_n$ or $n\,!\,{}^{n-1}C_{r-1}$ according as blank group are or are not admissible.

(2) The number of ways in which n different things can be distributed into r different group is

$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - .... + (-1)^{n-1}\,{}^nC_{r-1}$ or Coefficient of x^n is $n\,!\,(e^x - 1)^r$

Here blank groups are not allowed.

(3) Number of ways in which $m \times n$ different objects can be distributed equally among n persons (or numbered groups) = (number of ways of dividing into groups) × (number of groups) ! = $\frac{(mn)!n!}{(m!)^n n!} = \frac{(mn)!}{(m!)^n}$.

Case II : (1) The number of ways in which $(m+n)$ different things can be divided into two groups which contain m and n things respectively is, ${}^{m+n}C_m . {}^nC_n = \frac{(m+n)!}{m!n!}, m \neq n$.

Corollary: If $m = n$, then the groups are equal size. Division of these groups can be given by two types.

Type I : If order of group is not important : The number of ways in which $2n$ different things can be divided equally into two groups is $\frac{(2n)!}{2!(n!)^2}$

Type II : If order of group is important : The number of ways in which $2n$ different things can be divided equally into two distinct groups is $\frac{(2n)!}{2!(n!)^2} \times 2! = \frac{2n!}{(n!)^2}$

(2) The number of ways in which $(m + n + p)$ different things can be divided into three groups which contain m, n and p things respectively is ${}^{m+n+p}C_m . {}^{n+p}C_n . {}^pC_p = \frac{(m+n+p)!}{m!n!p!}, m \neq n \neq p$

Corollary: If $m = n = p$, then the groups are equal size. Division of these groups can be given by two types.

Type I : If order of group is not important : The number of ways in which $3p$ different things can be divided equally into three groups is $\frac{(3p)!}{3!(p!)^3}$

Type II : If order of group is important : The number of ways in which $3p$ different things can be divided equally into three distinct groups is $\frac{(3p)!}{3!(p!)^3} t3! = \frac{(3p)!}{(p!)^3}$

Note :
- ❑ If order of group is not important : The number of ways in which mn different things can be divided equally into m groups is $\frac{mn!}{(n!)^m m!}$
- ❑ If order of group is important: The number of ways in which mn different things can be divided equally into m distinct groups is $\frac{(mn)!}{(n!)^m m!} \times m! = \frac{(mn)!}{(n!)^m}$.

Example: 37 In how many ways can 5 prizes be distributed among four students when every student can take one or more prizes

[BIT Ranchi 1990; Rajasthan PET 1988, 97]

(a) 1024 (b) 625 (c) 120 (d) 60

Solution: (a) The required number of ways = $4^5 = 1024$ [since each prize can be distributed by 4 ways]

Example: 38 The number of ways in which 9 persons can be divided into three equal groups is

(a) 1680 (b) 840 (c) 560 (d) 280

Solution: (d) Total ways = $\frac{9!}{(3!)^3} = \frac{9\times8\times7\times6\times5\times4}{3\times2\times3\times2\times3\times2} = 280$.

Example: 39 The number of ways dividing 52 cards amongst four players equally, are **[IIT 1979]**

(a) $\frac{52!}{(13!)^4}$ (b) $\frac{52!}{(13!)^2 4!}$ (c) $\frac{52!}{(12!)^4 4!}$ (d) None of these

Solution: (a) Required number of ways = ${}^{52}C_{13}\times{}^{39}C_{13}\times{}^{26}C_{13}\times{}^{13}C_{13} = \frac{52!}{39!13!}\times\frac{39!}{26!13!}\times\frac{26!}{13!13!}\times\frac{13!}{13!} = \frac{52!}{(13!)^4}$.

Example: 40 A question paper is divided into two parts A and B and each part contains 5 questions. The number of ways in which a candidate can answer 6 questions selecting at least two questions from each part is

(a) 80 (b) 100 (c) 200 (d) None of these

Solution: (c) The number of ways that the candidate may select

2 questions from A and 4 from B = ${}^5C_2\times{}^5C_4$; 3 questions form A and 3 from B = ${}^5C_3\times{}^5C_3$

4 questions from A and 2 from B = ${}^5C_4\times{}^5C_2$. Hence total number of ways are 200.

5.14 Derangement

Any change in the given order of the things is called a derangement.

If n things form an arrangement in a row, the number of ways in which they can be deranged so that no one of them occupies its original place is $n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+......+(-1)^n.\frac{1}{n!}\right)$.

Example: 41 There are four balls of different colours and four boxes of colurs same as those of the balls. The number of ways in which the balls, one in each box, could be placed such that a ball doesn't go to box of its own colour is **[IIT 1992]**

(a) 8 (b) 7 (c) 9 (d) None of these

Solution: (c) Number of derangement are = $4!\left\{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right\} = 12-4+1=9$.

(Since number of derangements in such a problem is given by $n!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}......+(-1)^n\frac{1}{n!}\right\}$.

5.15 Some Important Results for Geometrical Problems

(1) Number of total different straight lines formed by joining the n points on a plane of which m ($< n$) are collinear is ${}^nC_2 - {}^mC_2 + 1$.

(2) Number of total triangles formed by joining the n points on a plane of which m ($< n$) are collinear is ${}^nC_3 - {}^mC_3$.

(3) Number of diagonals in a polygon of n sides is ${}^nC_2 - n$.

(4) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed is ${}^mC_2\times{}^nC_2$ i.e $\frac{mn(m-1)(n-1)}{4}$

(5) Given n points on the circumference of a circle, then

(i) Number of straight lines = ${}^nC_2$ (ii) Number of triangles = ${}^nC_3$ (iii) Number of quadrilaterals = ${}^nC_4$.

(6) If n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then the number of part into which these lines divide the plane is = $1+\Sigma n$.

(7) Number of rectangles of any size in a square of $n\times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(8) In a rectangle of $n\times p$ $(n<p)$ number of rectangles of any size is $\frac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n}(n+1-r)(p+1-r)$.

Example: 42 The number of diagonals in a octagon will be **[MP PET 1984; Pb. CET 1989, 2000]**

(a) 28 (b) 20 (c) 10 (d) 16

Solution: (b) Number of diagonals = ${}^8C_2-8$ = $28-8$ = 20.

Example: 43 The number of straight lines joining 8 points on a circle is

(a) 8 (b) 16 (c) 24 (d) 28

Solution: (d) Number of straight line = ${}^8C_2$ = 28.

Example: 44 The number of triangles that can be formed by choosing the vertices from a set of 12 points, seven of which lie on the same straight line, is **[Roorkee 1989, 2000; BIT Ranchi 1989; MP PET 1995; Pb. CET 1997; DCE 20**

(a) 185 (b) 175 (c) 115 (d) 105

Solution: (a) Required number of ways = ${}^{12}C_3-{}^7C_3$ = 220 – 35 = 185.

Example: 45 Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. The number of (i) straight lines (ii) triangles which can be formed by joining them

(i) (a) 140 (b) 142 (c) 144 (d) 146

(ii) (a) 816 (b) 806 (c) 800 (d) 750

Solution: (c, b) Out of 18 points, 5 are collinear

(i) Number of straight lines = ${}^{18}C_2-{}^5C_2+1=153-10+1=144$

(ii) Number of triangles $={}^{18}C_3-{}^5C_3=816-10=806$.

5.16 Multinomial Theorem

Let $x_1,x_2,......,x_m$ be integers. Then number of solutions to the equation $x_1+x_2+......+x_m=n$ .....(i)

Subject to the condition $a_1\le x_1\le b_1, a_2\le x_2\le b_2,......., a_m\le x_m\le b_m$ .....(ii)

is equal to the coefficient of x^n in

$(x^{a_1}+x^{a_1+1}+......+x^{b_1})(x^{a_2}+x^{a_2+1}+.....+x^{b_2})......(x^{a_m}+x^{a_{m+1}}+.....+x^{b_m})$ ......(iii)

This is because the number of ways, in which sum of *m* integers in (i) equals *n*, is the same as the number of times x^n comes in (iii).

(1) **Use of solution of linear equation and coefficient of a power in expansions to find the number of ways of distribution :** (i) The number of integral solutions of $x_1+x_2+x_3+......+x_r=n$ where $x_1\ge 0, x_2\ge 0,......x_r\ge 0$ is the same as the number of ways to distribute *n* identical things among *r* persons.

This is also equal to the coefficient of x^n in the expansion of $(x^0+x^1+x^2+x^3+......)^r$

= coefficient of x^n in $\left(\frac{1}{1-x}\right)^r$ = coefficient of x^n in $(1-x)^{-r}$

$= \text{coefficient of } x^n \text{ in } \left\{1 + rx + \frac{r(r+1)}{2!}x^2 + ... + \frac{r(r+1)(r+2)......(r+n-1)}{n!}x^n + ...\right.$

$= \frac{r(r+1)(r+2)....(r+n-1)}{n!} = \frac{(r+n-1)!}{n!(r-1)!} = {}^{n+r-1}C_{r-1}$

(ii) The number of integral solutions of $x_1 + x_2 + x_3 + ... + x_r = n$ where $x_1 \geq 1, x_2 \geq 1, ...x_r \geq 1$ is same as the number of ways to distribute n identical things among r persons each getting at least 1. This also equal to the coefficient of x^n in the expansion of $(x^1 + x^2 + x^3 + ...)^r$

$= \text{coefficient of } x^n \text{ in } \left(\frac{x}{1-x}\right)^r = \text{coefficient of } x^n \text{ in } x^r(1-x)^{-r}$

$= \text{coefficient of } x^n \text{ in } x^r\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + ... + \frac{r(r+1)(r+2).....(r+n-1)}{n!}x^n + ...\right\}$

$= \text{coefficient of } x^{n-r} \text{ in } \left\{1 + rx + \frac{r(r+1)}{2!}x^2 + ... + \frac{r(r+1)(r+2).....(r+n-1)}{n!}x^n + ...\right\}$

$= \frac{r(r+1)(r+2)......(r+n-r-1)}{(n-r)!} = \frac{r(r+1)(r+2).....(n-1)}{(n-r)!} = \frac{(n-1)!}{(n-r)!(r-1)!} = {}^{n-1}C_{r-1}$.

Example: 46 A student is allowed to select utmost n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select one book is 63, then the value of n is **[IIT 1987; Rajasthan PET 1999]**

(a) 2 (b) 3 (c) 4 (d) None of these

Solution: (b) Since the student is allowed to select utmost n books out of $(2n+1)$ books. Therefore in order to select one book he has the choice to select one, two, three,......., n books.

Thus, if T is the total number of ways of selecting one book then $T = {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n = 63$.

Again the sum of binomial coefficients

${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + ... + {}^{2n+1}C_{2n+1} = (1+1)^{2n+1} = 2^{2n+1}$

or, ${}^{2n+1}C_0 + 2({}^{2n-1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$

$\Rightarrow 1 + 2(T) + 1 = 2^{2n+1} \Rightarrow 1 + T = \frac{2^{2n+1}}{2} = 2^{2n} \Rightarrow 1 + 63 = 2^{2n} \Rightarrow 2^6 = 2^{2n} \Rightarrow n = 3$.

Example: 47 If x, y and r are positive integers, then ${}^xC_r + {}^xC_{r-1}{}^yC_1 + {}^xC_{r-2}\,{}^yC_2 + ... + {}^yC_r =$

[Karnataka CET 1993; Rajasthan PET 2001]

(a) $\frac{x!y!}{r!}$ (b) $\frac{(x+y)!}{r!}$ (c) ${}^{x+y}C_r$ (d) ${}^{xy}C_r$

Solution: (c) The result ${}^{x+y}C_r$ is trivially true for $r = 1, 2$ it can be easily proved by the principle of mathematical induction that the result is true for r also.

5.17 Number of Divisors

Let $N = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}......p_k^{\alpha_k}$, where $p_1, p_2, p_3, ...p_k$ are different primes and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k$ are natural numbers then :

(1) The total number of divisors of N including 1 and N is = $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)....(\alpha_k + 1)$

(2) The total number of divisors of N excluding 1 and N is $= (\alpha_1+1)(\alpha_2+1)(\alpha_3+1).....(\alpha_k+1)-2$

(3) The total number of divisors of N excluding 1 or N is $= (\alpha_1+1)(\alpha_2+1)(\alpha_3+1).....(\alpha_k+1)-1$

(4) The sum of these divisors is $=(p_1^0+p_2^1+p_3^2+......+p_1^{\alpha_1})(p_2^0+p_2^1+p_2^2+...+p_2^{\alpha_2}).....(p_k^0+p_k^1+p_k^2+....+p_k^{\alpha_k})$

(5) The number of ways in which N can be resolved as a product of two factors is

$$\begin{cases}\frac{1}{2}(\alpha_1+1)(\alpha_2+1)....(\alpha_k+1), \text{If } N \text{ is not a perfect square}\\ \frac{1}{2}[(\alpha_1+1)(\alpha_2+1).....(\alpha_k+1)+1], \text{If } N \text{ is a perfect square}\end{cases}$$

(6) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or co-prime) to each other is equal to 2^{n-1} where n is the number of different factors in N.

Important Tips

- *All the numbers whose last digit is an even number 0, 2, 4, 6 or 8 are divisible by 2.*
- *All the numbers sum of whose digits are divisible by 3,is divisible by 3 e.g. 534. Sum of the digits is 12, which are divisible by 3, and hence 534 is also divisible by 3.*
- *All those numbers whose last two-digit number is divisible by 4 are divisible by 4 e.g. 7312, 8936, are such that 12, 36 are divisible by 4 and hence the given numbers are also divisible by 4.*
- *All those numbers, which have either 0 or 5 as the last digit, are divisible by 5.*
- *All those numbers, which are divisible by 2 and 3 simultaneously, are divisible by 6. e.g., 108, 756 etc.*
- *All those numbers whose last three-digit number is divisible by 8 are divisible by 8.*
- *All those numbers sum of whose digit is divisible by 9 are divisible by 9.*
- *All those numbers whose last two digits are divisible by 25 are divisible by 25 e.g., 73125, 2400 etc.*

Example: 48 The number of divisors of 9600 including 1 and 9600 are

(a) 60 (b) 58 (c) 48 (d) 46

Solution: (c) Since 9600 = $2^7 \times 3^1 \times 5^2$

Hence number of divisors = $(7+1)(1+1)(2+1)=48$.

Example: 49 Number of divisors of $n=38808$ (except 1 and n) is

(a) 70 (b) 68 (c) 72 (d) 74

Solution: (a) Since 38808 = 8 × 4851 = 8 × 9 × 539 = 8 × 9 × 7 × 7 × 11 = $2^3 \times 3^2 \times 7^2 \times 11$

So, number of divisors = (3 + 1) (2 + 1) (2 + 1) (1 + 2) – 2 = 72 – 2 = 70.

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